Close Wed: HW_5A,5B,5C (7.1,7.2,7.3)
Office Hours: 1:30-3:00 in Smith 309

## <u>**7.2 Trig Integrals**</u> (*continued*)

*Entry Task*: Fill in the blanks

| **Square Identities** |
|---|
| $\sin^2(x) = 1 - \cos^2(x)$ |
| $\cos^2(x) = 1 - \sin^2(x)$ |
| $\tan^2(x) = \sec^2(x) - 1$ |
| $\sec^2(x) = 1 + \tan^2(x)$ |
| **Half Angle Identities** |
| $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ |
| $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ |
| $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$ |

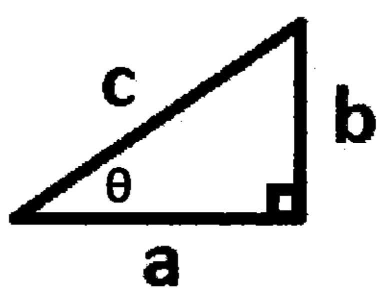
What are these in terms of a, b, and c?

$\sin(\theta) =$ $b/c$

$\cos(\theta) =$ $a/c$

$\tan(\theta) =$ $b/a$

$\sec(\theta) =$ $c/a$




Integrate

$$\int \sin^3(x)\cos^5(x)dx$$

$$\int \sin^2(x)\cos^5(x)\sin(x)dx$$

$$\int (1-\cos^2(x))\cos^5(x)\sin(x)dx$$

$$\int (1-u^2)u^5(-1)du$$

$u = \cos(x)$

$du = -\sin(x)dx$

$(-1)du = \sin(x)dx$

$$-\int u^5 - u^7 du$$

$$-\frac{1}{6}u^6 + \frac{1}{8}u^8 + c$$

$$\boxed{-\frac{1}{6}\cos^6(x) + \frac{1}{8}\cos^8(x) + C}$$

## Integrals involving sin(x) and cos(x)

**Case 1 (*cos(x) or sin(x) has odd power*)**

- Separate one from the odd power. (*i.e.* pull out one sin(x) or cos(x))
- Use $\sin^2(x) = 1 - \cos^2(x)$
  $\cos^2(x) = 1 - \sin^2(x)$
  (get rest in term of the other)
- Use u-substitution.

**Case 2 (*both sin(x),cos(x) even powers*)**

- Use

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

over and over again until you can integrate.

*Example*:

$$\int \cos^4(x)\,dx$$

$$= \int \cos^2(x)\cos^2(x)\,dx$$

$$= \int \tfrac{1}{2}(1+\cos(2x))\tfrac{1}{2}(1+\cos(2x))\,dx$$

$$= \tfrac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\,dx$$

$$= \tfrac{1}{4}\int 1 + 2\cos(2x) + \tfrac{1}{2}(1+\cos(4x))\,dx$$

$$= \tfrac{1}{4}\int \tfrac{3}{2} + 2\cos(2x) + \tfrac{1}{2}\cos(4x)\,dx$$

$$= \tfrac{1}{4}\left[\tfrac{3}{2}x + \sin(2x) + \tfrac{1}{8}\sin(4x)\right] + C$$

$$= \boxed{\tfrac{3}{8}x + \tfrac{1}{4}\sin(2x) + \tfrac{1}{32}\sin(4x) + C}$$

## Integrals involving tan(x) and sec(x)

**Case 3 (sec*(x) has an even power*)**

- Separate out $\sec^2(x)$
- Use $\sec^2(x) = 1 + \tan^2(x)$
  (get rest in terms of $\tan(x)$)
- Use $u = \tan(x)$

*Example*: $\int \tan^5(\mathrm{x})\sec^4(\mathrm{x})dx$

$$\int \tan^5(x)\sec^2(x)\sec^2(x)dx$$

$$\int \tan^5(x)(1+\tan^2(x))\sec^2(x)dx$$

$$\int u^5(1+u^2)\,du$$

$u = \tan(x)dx$

$du = \sec^2(x)dx$

$$\int u^5 + u^7 du$$

$$= \frac{1}{6}u^6 + \frac{1}{8}u^8 + C$$

$$= \boxed{\frac{1}{6}\tan^6(x) + \frac{1}{8}\tan^8(x) + C}$$

**Case 4 (tan*(x) has an odd power*)** AND AT LEAST ONE $\sec(x)$

- Separate out $\sec(x)\tan(x)$
- Use $\tan^2(x) = \sec^2(x) - 1$
  (get rest in terms of $\sec(x)$)
- Use $u = \sec(x)$

*Example*: $\int \tan^3(\mathrm{x})\sec(\mathrm{x})dx$

$$\int \tan^2(x)\sec(x)\tan(x)dx$$

$$\int (\sec^2(x) - 1)\sec(x)\tan(x)dx$$

$u = \sec(x)$

$du = \sec(x)\tan(x)dx$

$$\int u^2 - 1\, du$$

$$\frac{1}{3}u^3 - u + C$$

$$\boxed{\frac{1}{3}\sec^3(x) - \sec(x) + C}$$

And if you've tried the four cases and are stuck, here are things to try:

1. Rewrite in terms of sin(x) and cos(x).
2. Rewrite in terms of sec(x) and tan(x).
3. Try using trig identities.

Ex) $\int \frac{\cos(x) + \sin(2x)}{\sin(x)}\,dx$ to start

$= \int \frac{\cos(x) + 2\sin(x)\cos(x)}{\sin(x)}\,dx$

$\int \frac{\tan(x)\cos(x)}{\sec(x)}\,dx = \int \frac{\frac{\sin(x)}{\cos(x)}\cos(x)}{(1/\cos(x))}\,dx$

$= \int \sin(x)\cos(x)\,dx$

There are still a few "holes".
Namely, when there is one tan(x) or an odd power on sec(x). You can quote these (proof in the book):

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C$$

## 7.3 Trigonometric Substitution

*Goal*: Develop a method to evaluate integrals involving expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$

| CASE | SUBSTITUTION |
|---|---|
| $a^2 - x^2$ | $x = a\sin(\theta)$, $-\pi/2 \le \theta \le \pi/2$ |
| $a^2 + x^2$ | $x = a\tan(\theta)$, $-\pi/2 < \theta < \pi/2$ |
| $x^2 - a^2$ | $x = a\sec(\theta)$, $0 \le \theta < \pi/2$, $\pi \le \theta < 3\pi/2$ |

*Example:*

1. $\displaystyle\int \frac{x^3}{\sqrt{4 - x^2}}\,dx$

$\boxed{x = 2\sin\theta}$

$dx = 2\cos\theta\, d\theta$

$$= \int \frac{8\sin^3\theta}{\sqrt{4 - 4\sin^2\theta}}\, 2\cos\theta\, d\theta$$

$$= \int \frac{8\sin^3\theta}{\cancel{2\cos\theta}}\, \cancel{2\cos\theta}\, d\theta$$

$$= 8\int \sin^3\theta\, d\theta$$

KEY NOTE

$$4 - 4\sin^2\theta = 4(1 - \sin^2\theta)$$
$$= 4\cos^2\theta$$
$$\Rightarrow \sqrt{4 - 4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

$$8\int \sin^2\theta \sin\theta\, d\theta$$

$$8\int (1 - \cos^2\theta)\sin\theta\, d\theta$$

$$8\int (1 - u^2)(-1)\,du$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

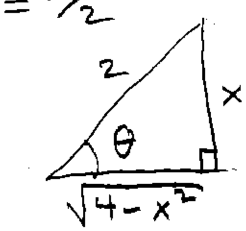
$(-1)du = \sin\theta\, d\theta$

$$-8u + \frac{8}{3}u^3 + C$$

$$-8\cos\theta + \frac{8}{3}\cos^3\theta + C$$

$$-8\,\frac{\sqrt{4 - x^2}}{2} + \frac{8}{3}\left(\frac{\sqrt{4 - x^2}}{2}\right)^3 + C$$

$\sin\theta = x/2$




$\cos\theta = \dfrac{\sqrt{4 - x^2}}{2}$

$$\boxed{-4\sqrt{4 - x^2} + \frac{1}{3}(4 - x^2)^{3/2} + C}$$

Trigonometric Substitution Method:

A) Substitute, don’t forget dx = ??dθ.
   Simplify (eliminate root)

B) Use 7.2 methods for trig integrals.

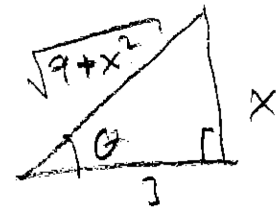
C) Draw a triangle and return to x.

$\tan\theta = x/3$

2. $\int \sqrt{9+x^2}\,dx$

$\boxed{x = 3\tan\theta}$

$dx = 3\sec^2\theta\, d\theta$




$\sec\theta = \dfrac{\sqrt{9+x^2}}{3}$

$$\int \underbrace{\sqrt{9+9\tan^2\theta}}_{3\sec\theta}\; 3\sec^2\theta\, d\theta$$

$$9\int \sec^3\theta\, d\theta$$

$$\frac{9}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$

$$\frac{9}{2}\left(\frac{\sqrt{9+x^2}}{3}\,\frac{x}{3} + \ln\left|\frac{\sqrt{9+x^2}}{3} + x/3\right|\right) + C$$

$$= \frac{1}{2}x\sqrt{9+x^2} + \frac{9}{2}\left(\ln|\sqrt{9+x^2} + x| - \ln(3)\right) + C$$

$$= \frac{1}{2}x\sqrt{9+x^2} + \frac{9}{2}\ln|\sqrt{9+x^2} + x| + D$$

3. $\int \frac{\sqrt{x^2 - 16}}{x} dx$

$\boxed{x = 4\sec\theta}$

$dx = 4\sec\theta\tan\theta\, d\theta$

$\sqrt{16(\sec^2\theta - 1)}$

$= \sqrt{16\tan^2\theta}$

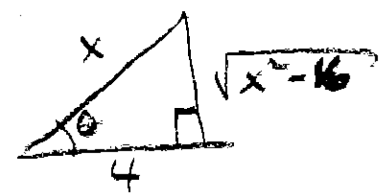
$= 4\tan\theta$

$$\int \frac{\sqrt{16\sec^2\theta - 16}}{\cancel{4\sec\theta}} \cancel{4\sec\theta}\tan\theta\, d\theta$$

$$\int 4\tan\theta \tan\theta\, d\theta$$

$$= 4\int \tan^2\theta\, d\theta$$

$\sec\theta = x/4$




$\theta = \sec^{-1}(x/4)$

$$= 4\int \sec^2\theta - 1\, d\theta$$

$$= 4(\tan\theta - \theta) + C$$

$$= 4\tan\theta - 4\theta + C$$

$$= 4\frac{\sqrt{x^2-16}}{4} - 4\sec^{-1}(x/4) + C = \boxed{\sqrt{x^2-16} - 4\sec^{-1}(x/4) + C}$$

Completing the Square:

$$\sqrt{ax^2 + \boldsymbol{bx} + c}$$

If you encounter a "**middle term**" (like **bx** above), then complete the square.

*Example*: $\sqrt{64 - 24x - 4x^2}$

i) **Factor out the "a".**

$$\sqrt{4(16 - 6x - x^2)} = 2\sqrt{16 - 6x - x^2}$$

ii) **Add/subtract "half-middle squared"**

Half of middle = (-6)/2 = -3

Squared = $(-3)^2$ = 9

$$2\sqrt{16 + 9 - 9 - 6x - x^2}$$

iii) **Factor the perfect square**

$$2\sqrt{25 - (x + 3)^2}$$

iv) **Check your work!!!!!**

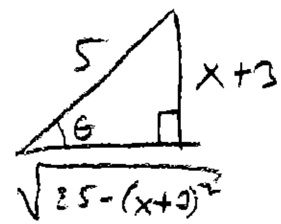
*Example*:

$$\int \frac{x}{\sqrt{64-24x-4x^2}}\,dx$$

$$= \int \frac{x}{2\sqrt{25-(x+3)^2}}\,dx$$

$\boxed{x+3 = 5\sin\theta} \to x = 5\sin\theta - 3$ $\qquad \sin\theta = \frac{x+3}{5}$

$dx = 5\cos\theta\,d\theta$




$$= \frac{1}{2}\int \frac{(5\sin\theta - 3)}{\sqrt{25-25\sin^2\theta}}\,5\cos\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{(5\sin\theta - 3)}{\cancel{5\cos\theta}}\,\cancel{5\cos\theta}\,d\theta$$

$$= \frac{1}{2}\left(-5\cos\theta - 3\theta\right) + C$$

$$= -\frac{5}{2}\cos\theta - \frac{3}{2}\theta + C$$

$$= \boxed{-\frac{\cancel{5}}{2}\,\frac{\sqrt{25-(x+3)^2}}{\cancel{5}} - \frac{3}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + C}$$